Received: *December 18, 1998*, Accepted: *January 6, 1999*

Vacua, Propagators, and Holographic Probes in AdS/CFT

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Abstract: In this paper we investigate the relation between the bulk and boundary in AdS/CFT. We first discuss the relation between the Poincaré and the global vacua, and then study various probes of the bulk from the boundary theory point of view. We derive expressions for retarded propagators and note that objects in free fall look like expanding bubbles in the boundary theory. We also study several Yang-Mills theory examples where we investigate thermal screening and confinement using propagators. In the case of confinement we also calculate the profile of a flux tube and provide an alternative derivation of the tension.

Keywords: D-branes, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

After the discovery of the AdS/CFT correspondence [1] (see also [2, 3]) and its more elaborated formulation in [4, 5], numerous supergravity calculations have been used to learn more about supersymmetric Yang-Mills theories. Particularly interesting is the calculation of Wilson loops in [6, 7] which have also been generalized to finite temperature [8]–[12]. Ultimately, one hopes to be able to use the correspondence to study also non supersymmetric Yang-Mills theories, preliminary steps in this direction have been taken in [13, 14]. In particular there exist calculations of glueball masses that agree reasonably well with lattice calculations [15]–[18]. It is however far from clear how to connect these models with real QCD in a rigorous way [19, 20].

In [21, 22] the relation between scale in the boundary and radial position in the bulk was studied in some detail. This is especially interesting from the point of view of holography, i.e. the capability of the boundary theory to fully encode the bulk theory. (Note that this can be expected only if the bulk theory includes gravity, while the use of a bulk theory to describe a boundary theory can be expected to be more general). In this paper we will continue the investigation of this correspondence.

In section 2 we clarify some issues concerning vacua defined using various coordinate systems. Next we derive exact retarded propagators that can be used to calculate boundary fields for time dependent bulk configurations. In section 4 we proceed by considering various Yang-Mills theory examples. We derive the screening length and the $F^2_{\mu\nu}$ expectation value at finite temperature using WKB techniques. For the confining case we calculate the shape of a flux tube and verify that the integrated energy density gives a tension that reproduces the confining potential calculated using other means.

2. The global and the Poincaré vacuum

A basic starting point in field theory is the vacuum state. In curved space such as the AdS space the appropriate choice is often non-trivial and has a great impact on the physical picture. For example, in investigating the thermodynamics of black holes it is crucial to impose a correct initial vacuum state to see the Hawking radiation. An often used vacuum choice in the AdS/CFT literature is the vacuum corresponding to the Poincaré coordinate system, the Poincaré vacuum. For example, the Poincaré vacuum is the chosen initial vacuum state in studies of thermodynamics (see *e.g.* [23, 24, 25, 26]) of BTZ black holes [27, 28]. A slightly puzzling issue is that the Poincaré coordinates cover only a part of the AdS manifold so there is a coordinate horizon. It would seem preferable to use coordinates that cover the whole manifold, the global coordinates, and use the associated global vacuum state as a starting point. It is therefore important to understand if there is a difference in the two vacuum choices, and what the difference will be.

In this section we study the relation between the global and the Poincaré vacuum. First we define the vacua in the framework of quantized free bulk fields and the corresponding operators in the boundary theory, and make a comparison between the states in the Poincaré and global coordinates. We then move into a more general level and study the Virasoro generators in the Poincaré and global coordinates, and relate the two vacua into each other using the transformation properties of the Virasoro generators. By the bulk/boundary correspondence, it should not matter whether the states are compared using their bulk theory or their boundary theory representation, since both represent the same Hilbert space. We will therefore mostly focus on the representation of the states in the boundary theory.

2.1 AdS coordinates

In this section we collect some results concerning various coordinate systems for AdS. We will focus on AdS_3, but the extension of the results to the general case should be obvious.

AdS_3 can be defined as the submanifold of $R^{2,2}$ defined by the equation

$$U^2 + V^2 - X^2 - Y^2 = 1\,, \tag{2.1}$$

where U,V,X and Y parameterize the embedding $R^{2,2}$.

In the following we will use three coordinate systems, namely global (τ, μ, θ), Poincaré (t, x_0, x) and BTZ $(\tilde{t}, \phi, r)$ defined through

$$\begin{aligned} U &= \cosh\mu\cos\tau = \frac{1}{2x_0}(1 + x^2 + x_0^2 - t^2) = r\cosh(r_+\phi - r_- t)\,, \\ V &= \cosh\mu\sin\tau = \frac{t}{x_0} = \sqrt{r^2-1}\sinh(-r_-\phi + r_+ t)\,, \\ X &= \sinh\mu\sin\theta = \frac{x}{x_0} = r\sinh(r_+\phi - r_- t)\,, \\ Y &= \sinh\mu\cos\theta = \frac{1}{2x_0}(1 - x^2 - x_0^2 + t^2) = \sqrt{r^2-1}\cosh(-r_-\phi + r_+ t)\,. \end{aligned} \tag{2.2}$$

The BTZ coordinates parameterize only a patch within the Poincaré space which is also only part of the whole AdS_3 space spanned by the global coordinates. If the identification $\phi \equiv \phi + 2\pi$ is made in BTZ coordinates then the space becomes a black hole and the parameters r_+ and r_- represent the radii of the outer and inner horizons.

The boundary of AdS space is at $\mu \to \infty$, $x_0 \to 0$, $r \to \infty$ in the different coordinates. Near the boundary the coordinate transformations look like conformal transformations between boundary variables. It is easy to see that the conformal transformations are

$$\begin{aligned} z &= \tan\left(\frac{w}{2}\right) = \tanh\left((r_+ - r_-)\frac{\phi_+}{2}\right), \\ \bar{z} &= \tan\left(\frac{\bar{w}}{2}\right) = \tanh\left((r_+ + r_-)\frac{\phi_-}{2}\right), \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} z &= t + x\,, \qquad & w &= \tau + \theta\,, \qquad & \phi_+ &= \tilde{t} + \phi\,, \\ \bar{z} &= t - x\,, \qquad & \bar{w} &= \tau - \theta\,, \qquad & \phi_- &= \tilde{t} - \phi\,. \end{aligned} \tag{2.4}$$

2.2 Quantization of the bulk fields and boundary operators

We begin with a short review of the operator quantization scheme in the bulk and boundary theory in Minkowski signature, following the references [21][22]. A free bulk field is promoted to a quantized operator through an expansion

$$\Phi = \sum_k a_k \phi_k(r, \vec{x}) \ + \ \text{h.c.}\,, \tag{2.5}$$

where ϕ_k are the normalizable modes in the bulk, in the chosen coordinate system, and a_k are the associated annihilation operators. Here r denotes the radial coordinate, and $\vec{x}$ labels the boundary coordinates.

The vacuum $|0\rangle$ is defined as the state that satisfies

$$a_k|0\rangle = 0\,, \tag{2.6}$$

for all annihilation operators. If we use the Poincaré coordinates, we call the resulting vacuum the Poincaré vacuum, similarly the global coordinates define the global vacuum.

The boundary operator $\mathcal{O}$ coupling to the bulk field Φ has a similar operator expansion

$$\mathcal{O} = \sum_k b_k \tilde{\phi}_k(\vec{x}) \ + \ \text{h.c.}\,, \tag{2.7}$$

where $\tilde{\phi}_k$ are the boundary data extracted from the asymptotic behavior of the normalizable bulk modes near the boundary:

$$\phi_k \sim r^{2h_+} \tilde{\phi}_k(\vec{x})\,. \tag{2.8}$$

By the bulk/boundary correspondence, the vacuum defined by

$$b_k|0\rangle = 0 \tag{2.9}$$

corresponds to the same state in the abstract Hilbert space of the theory as the vacuum defined using the bulk annihilation operators a_k. We have only changed the representation of the Hilbert space from bulk to boundary theory language.

Let us now take a closer look into the boundary theory in the two coordinate systems. The boundary is the boundary of the covering space of AdS_3 (in this section we will focus on 2+1 dimensions). It is an infinite cylinder and can be covered by the global coordinates $\tau \in R, \theta \in [0, 2\pi]$ with the periodic identification $\theta \sim \theta + 2\pi$.

In global coordinates the mode expansion has a discrete spectrum,

$$\begin{aligned} \mathcal{O} &= \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} b^{(g)}_{n,l} e^{-i\omega_{n,l}\tau + il\theta} \ + \ \text{h.c.} \\ &= \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} b^{(g)}_{h_{n,l},\bar{h}_{n,l}} \ e^{-ih_{n,l}w - i\bar{h}_{n,l}\bar{w}} \ + \ \text{h.c.}\,, \end{aligned} \tag{2.10}$$

where we used the null coordinates $w = \tau + \theta$, $\bar{w} = \tau - \theta$. The weights $h_{n,l}, \bar{h}_{n,l}$ are related to the mode frequencies $\omega_{n,l}$ and the angular momenta l by

$$\begin{aligned} h_{n,l} &= \frac{1}{2}(\omega_{n,l} - l)\,, \\ \bar{h}_{n,l} &= \frac{1}{2}(\omega_{n,l} + l) \end{aligned} \tag{2.11}$$

and the frequencies $\omega_{n,l}$ are given by

$$\omega_{n,l} = 1 + \nu + |l| + 2n \tag{2.12}$$

and $\nu = \sqrt{1 + m^2\Lambda^2}$.

The global vacuum is given by

$$b^{(g)}_{n,l}|0\rangle_g = 0 \tag{2.13}$$

and the "one particle states" are given by

$$|n,l\rangle = b^{\dagger(g)}_{n,l}|0\rangle_g\,. \tag{2.14}$$

The latter transform according to the (elliptic) representation of SL(2,R) in the global coordinates [29, 30, 23, 21, 31].

The Poincaré coordinates cover only a finite diamond shaped patch $w \in [-\pi,\pi]$, $\bar{w} \in [-\pi,\pi]$ of the infinite cylindrical boundary of CAdS_3. The mode spectrum is continuous, and the operator expansion takes the form

$$\begin{aligned}\mathcal{O} &= \int_0^\infty d\omega \int_{-\infty}^\infty dk\; b^{(P)}_{\omega,k} e^{-i\omega t + ikx} \;+\; \text{h.c.} \\ &= \int dh \int d\bar{h}\; b^{(P)}_{h,\bar{h}} e^{-ihz - i\bar{h}\bar{z}} \;+\; \text{h.c.}\end{aligned} \tag{2.15}$$

in the null coordinates $z = t + x, \bar{z} = t - x$. The weights $(h, \bar{h})$ are related to the energy ω and the momentum k by

$$\begin{aligned}h &= \frac{1}{2}(\omega - k)\,, \\ \bar{h} &= \frac{1}{2}(\omega + k)\,.\end{aligned} \tag{2.16}$$

The Poincaré vacuum is given by

$$b^{(P)}_{\omega,k}|0\rangle_P = 0 \tag{2.17}$$

and the "one particle states" are given by

$$|\omega,k\rangle = b^{\dagger(P)}_{\omega,k}|0\rangle_p\,. \tag{2.18}$$

They transform according to the (parabolic) representation of SL(2,R) in the Poincaré coordinates [29, 30, 23, 21, 31].

2.3 The relation of the vacua

One might worry that since the global vacuum is defined over the whole infinite cylinder, a transformation to a single finite Poincaré patch will create a mixed state.

However, it turns out that if the parameter ν is an integer, the situation is simpler. One can check that in fact in this case the operator $\mathcal{O}$ becomes periodic or antiperiodic over a fundamental domain which is exactly the Poincaré diamond with periodic identifications $w \sim w + 2\pi, \bar{w} \sim \bar{w} + 2\pi$. In other words, we can view the operator $\mathcal{O}$ as being defined on a torus. If the parameter ν is an *odd* integer, the weights $h, \bar{h}$ are integer valued, and the operator $\mathcal{O}$ is periodic over the torus. If ν is an *even* integer, the weights $h, \bar{h}$ are half-integer valued and the operator $\mathcal{O}$ is antiperiodic over the torus.

The Poincaré patch does not quite cover a torus (one would need more than one patch). To manage with a single patch, we need to cut the torus open with

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infinitesimal cuts. If we begin with a global vacuum on the torus, the cuts in principle will induce a logarithmically divergent entropy due to the correlations lost across the cut [32, 33, 34]. However, this contribution can be ignored. Thus, after the cuts the system is still in a global vacuum state.

Now we can simply study how the modes transform under the coordinate transformation

$$z = \Lambda \tan(w/2)\,, \qquad \bar{z} = \Lambda \tan(\bar{w}/2)\,, \tag{2.19}$$

from the global coordinates to the Poincaré coordinates.[1] One can easily check that the Poincaré modes are analytic and bounded in the lower complex global time half-plane, so they correspond to positive energy modes in global coordinates.[2]

As an aside, we also note that the transformation from global to Poincaré coordinates is different from the transformation from Minkowski to Rindler coordinates in Minkowski space. The $\tau = 0$ global time slice coincides with the $t = 0$ Poincaré time slice, whereas a $t = 0$ Minkowski time slice is divided between two Rindler wedges.[3]

There is one subtlety which arises from mapping the finite region in global coordinates to an infinite region in Poincaré coordinates. The vacuum will develop a finite energy density. Subsequently the whole energy spectrum is shifted by the vacuum energy. We discuss this in more detail below.

2.4 The Virasoro generators

The discussion in the previous section was based on treating the operators as free fields. We will now study the vacua again in a more general framework and examine how Virasoro generators (corresponding to the generators of asymptotic isometries of AdS_3 [36]) are related in the two coordinate systems.

The Virasoro generators are a special case of conserved charges L_f corresponding to currents constructed from the energy momentum tensor:

$$L_f = \int dz\; T(z) f(z)\,, \tag{2.20}$$

where the integration is taken along an appropriate contour and $f(z)$ is a holomorphic function (see *e.g.* paper IV.2 in [37] for more discussion[4]). The charges satisfy the commutation relations

$$i[L_f, L_g] = L_h - \frac{a}{2}\int dz\; f'''(z) g(z)\,, \tag{2.21}$$

where $h = f'g - g'f$ and the last term is an additional c-number. In the radial quantization, one takes the integration contour to be a circle of fixed radius around the

[1] Incidentally, the same transformation is used to map the infinite Minkowski space to a finite Penrose diagram in 1+1 dimensions (see [35]). Thus the global coordinates are analogous to the Penrose coordinates and the Poincaré coordinates are analogous to the Minkowski coordinates.

[2] We thank Jorma Louko for emphasizing this to us.

[3] We thank I. Bengtsson and P. Kraus for pointing this out to us.

[4] In what follows, we will be somewhat cavalier and not address issues of convergence in the definition of the charges.

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origin, then the choice $f = -iz^{1-n}, g = -iz^{1-m}$ reproduces the Virasoro generators and the commutation relations of the Virasoro algebra with the central extension term where a is related to the central charge c.

In our case, for the global coordinates on the cylindrical boundary we choose to define the Virasoro generators as follows:

$$L_n^{(g)} = \int_{-\infty}^{\infty} dw \; e^{inw} T(w) \,, \tag{2.22}$$

similarly for the other null coordinate $\bar{w}$ we define $\bar{L}_n^{(g)}$. Recall that the mode spectrum is discrete in the global coordinates.

In the Poincaré coordinates, the spectrum is continuous, so we define a continuum of charges $L_\lambda^{(P)}$ (similarly $\bar{L}_\lambda^{(P)}$) by

$$L_\lambda^{(P)} = \int_{-\infty}^{\infty} dz \; e^{i\lambda z} T(z) \,. \tag{2.23}$$

Note that $L_{\lambda=0}^{(P)} + \bar{L}_{\lambda=0}^{(P)}$ is the Hamiltonian in the Poincaré coordinates.[5] Then, the commutation relations (2.21) imply that the $L_{\lambda\neq 0}^{(P)}$ charges act as raising and lowering operators on energy eigenstates:

$$[L_0^{(P)}, L_\lambda^{(P)}] = -\lambda L_\lambda^{(P)} \,. \tag{2.24}$$

Under the coordinate transformation from the global to Poincaré coordinates $w \to z(w)$, the stress tensor transforms as a dimension 2 operator

$$T^P(z) = \left(\frac{dw}{dz}\right)^2 T^g(w(z)) + \frac{c}{12} S(w,z) \,. \tag{2.25}$$

The first term in the RHS is the classical transformation, which at the quantum level is supplemented by the second term which arises from the conformal anomaly. $S(w,z)$ denotes the Schwarzian derivative.

Next, we work out the Schwarzian and the transformation rules for the individual charges L which we defined above. The results for the latter turn out to be:

$$L_{\lambda=0}^{(P)} = L_{n=0}^{(g)} + \frac{1}{2}(L_{n=-1}^{(g)} + L_{n=1}^{(g)}) - \frac{c}{24} \,, \tag{2.26}$$

$$L_{\lambda>0}^{(P)} = \sum_{n\geq -1} c_n(\lambda) L_n^{(g)} \,, \tag{2.27}$$

$$L_{\lambda<0}^{(P)} = \sum_{n\leq 1} \tilde{c}_n(\lambda) L_n^{(g)} \,, \tag{2.28}$$

where $c_n(\lambda), \tilde{c}_n(\lambda)$ are certain coefficients which can be expressed in terms of hypergeometric functions of λ. Note that the first relation (without the c-number term) is

[5]Compare this with the radial quantization on the Poincaré plane, where the Hamiltonian is $L_{-1} + \bar{L}_{-1}$, see *e.g.* [21].

equal to the relation between the time translation generating vector field $i\partial_z + i\partial_{\bar{z}}$ and the vector field representation of the SL(2,R) generators in the global coordinates.

Acting on the SL(2,R) invariant global vacuum $|0\rangle_g$ with the $L_\lambda^{(P)}$, we obtain the following results:

$$L^{(P)}_{\lambda=0}|0\rangle_g = -\frac{c}{24}\,, \tag{2.29}$$

$$L^{(P)}_{\lambda>0}|0\rangle_g = 0\,, \tag{2.30}$$

$$L^{(P)}_{\lambda<0}|0\rangle_g = \sum_{n\le -2} \tilde{c}_n(\lambda) L^g_n |0\rangle_g \neq 0\,. \tag{2.31}$$

The second equation above shows that $|0\rangle_g$ is the lowest energy state, because it is annihilated by all the lowering operators $L^{(P)}_{\lambda>0}$. The first equation tells that in the Poincaré coordinates the global vacuum state is seen to have a negative energy density. This can be understood as a shift of the vacuum energy density associated with the stretching of the finite area of the patch to an infinite area in the Poincaré coordinates. Consequently, the whole energy spectrum is shifted by the constant vacuum energy. This is a bit reminiscent of the Casimir effect, however in that case the restriction to the finite domain is obtained by imposing Dirichlet boundary conditions at the plates and the Casimir energy density depends on their separation. The last equation shows that $L^P_{\lambda<0}$ acts as a creation operator on the global vacuum.

From the bulk point of view we can formulate the result in the following way. It is easy to see that an observer at rest in Poincaré coordinates is accelerating with respect to an inertial observer using the global vacuum. This leads to the naive expectation that the Poincaré observer should see a thermal background, but as we have argued this is not the case. The background is not thermal but instead given by a shifted vacuum energy density. The acceleration of a Poincaré observer represents, however, a critical value such that any larger acceleration will result in a temperature as is the case for the vacuum defined by BTZ coordinates. See [38] for related results.

To summarize, the Poincaré vacuum state can be identified with the global vacuum state after the negative vacuum energy has been subtracted off. Hence in investigating quantum fields in AdS space, it does not make much of a difference whether one starts with a global or a Poincaré vacuum state. In particular, this result gives a more solid footing to the investigations of AdS black hole thermodynamics [23, 24, 25, 26].

3. Bulk to bulk propagators in AdS space

We now turn our attention to various propagators in the AdS space, these will be used later for the boundary theory description of different bulk probes.

The propagator for a scalar field is defined as a solution to the equation

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\partial_\nu G(x,x')\right) - m^2 G(x,x') = \frac{1}{\sqrt{g}}\delta(x-x')\,, \tag{3.1}$$

where $g^{\mu\nu}$ is the metric of the space and m the mass of the field. Also, x' is the position of the source and x the point where the field is measured. In the case of AdS_{d+1} space with Euclidean signature the metric is in Poincaré coordinates

$$ds^2 = \frac{1}{x_0^2}\left(dx_0^2 + dx_1^2 + \ldots dx_d^2\right). \tag{3.2}$$

In AdS, m^2 can be negative. A more useful parameter is given by:

$$\nu = \sqrt{\frac{d^2}{4} + m^2}\,, \tag{3.3}$$

which must be real to ensure stability, this imposes a lower bound $m^2 \geq -d^2/4$ for the mass [39, 40, 41]. The propagator is a function of the invariant distance $d(x, x')$ in AdS space, which in the Poincaré coordinates is defined by

$$\cosh(d(x,x')) = 1 + \frac{(x_0 - x_0')^2 + \ldots + (x_d - x_d')^2}{2x_0 x_0'}\,. \tag{3.4}$$

The solution of (3.1) can be found in [42] but we include it for completeness[6]:

$$\begin{aligned}
d &= 2m & G^{\nu}_{2m}(u) &= -\frac{1}{4\pi}\left(-\frac{1}{2\pi\sinh(u)}\frac{d}{du}\right)^{m-1}\frac{e^{-u\nu}}{\sinh(u)}\,, \\
d &= 2m+1 & G^{\nu}_{2m+1}(u) &= -\frac{1}{2\pi}\left(-\frac{1}{2\pi\sinh(u)}\frac{d}{du}\right)^{m} Q_{\nu-\frac{1}{2}}\left(\cosh(u)\right),
\end{aligned} \tag{3.5}$$

where $u = d(x, x')$ and Q is a Legendre function of the second kind [43]. These expressions are valid in any coordinate system provided we substitute the appropriate expression for the invariant distance u.

In the Minkowski case there are different propagators depending on the time boundary conditions, the most relevant ones are the Feynman and the retarded propagators. The Feynman propagator is the analytic continuation of the Euclidean one so it can be obtained from (3.5). The retarded propagator can be very different as is illustrated by the well-known example of 3+1 massless propagators in flat space. Recall also that in flat space the massless propagators are concentrated in the light cone in even dimensions. The Poincaré metric in Minkowski signature is given by:

$$ds^2 = \frac{1}{x_0^2}\left(dx_0^2 - dt^2 + dx_1^2 + \ldots dx_{d-1}^2\right). \tag{3.6}$$

The propagator depends on x_0', x_0, $t-t'$ and $r = \sqrt{(x_1 - x_1')^2 + \ldots + (x_{d-1} - x_{d-1}')^2}$. It satisfies the equation:

$$\begin{aligned}
\partial_{00}G + &\frac{1-d}{x_0}\partial_0 G - \partial_{tt}G + \partial_{rr}G + \frac{d-2}{r}\partial_r G + \frac{m^2}{x_0^2}G = \\
&= \frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{\frac{d-1}{2}}} x_0^{d-1} \frac{\delta(x_0 - x_0')\delta(t-t')\delta(r)}{r^{d-2}}\,.
\end{aligned} \tag{3.7}$$

[6]This expressions are equivalent to the one in [42] which uses an hypergeometric function.

In this equation the dimension d is just a parameter. A simple computation reveals a relation for propagators in different dimensions but for a fixed value of ν:

$$G^{\nu}_{d+2} = -\frac{1}{2\pi}\frac{x_0 x'_0}{r}\frac{d}{dr}G^{\nu}_{d}\,. \tag{3.8}$$

The only subtlety in deriving this relation is that in the right hand side of (3.7) the radial delta functions have different measures in different dimensions. Therefore, one must use the relation $r\delta'(r) = -\delta(r)$ for delta functions acting on functions regular at $r = 0$.

From (3.8) it follows that it is only necessary to obtain the propagators in AdS_{2+1} and AdS_{3+1}. The retarded propagator vanishes for $t < t'$ and for $t > t'$ it must be a linear combination of the solutions of the homogeneous equation. Among these solutions, we will only consider the normalizable ones. This amounts to a choice of a boundary condition in the boundary of AdS space which seems to be the most natural one given the formulation of the AdS/CFT correspondence in Lorentzian signature[21]. The normalizable modes can be written in terms of Bessel functions as [21]:

$$\Phi = e^{-i\omega t + ikx} x_0 J_\nu(qx_0)\,; \qquad w^2 = q^2 + k^2\,. \tag{3.9}$$

The linear combination is then chosen in such a way that

$$\partial_t G|_{t'\to t^+} - \partial_t G|_{t'\to t^-} = -\frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{\frac{d-1}{2}}}\, x_0^{d-1}\delta(x_0 - x'_0)\frac{\delta(r)}{r^{d-2}}\,. \tag{3.10}$$

Using the completeness relation of Bessel functions

$$\int_0^\infty dqq J_\nu(qx')J_\nu(qx) = \frac{\delta(x'-x)}{x}\,, \tag{3.11}$$

the propagator for $t > t'$ can be written as

$$\begin{aligned} G(x_0, x'_0, t-t', r) = -\frac{x_0 x'_0}{\pi}\int_0^\infty\int_0^\infty dk\, dq\, q \frac{\sin\left(\sqrt{q^2+k^2}(t-t')\right)}{\sqrt{q^2+k^2}} \times \\ \times \cos(kr) J_\nu(qx'_0) J_\nu(qx_0) \end{aligned} \tag{3.12}$$

Performing the k integral we obtain [43]

$$G(x_0, x'_0, t, r) = -\Theta(t-r)\frac{x_0 x'_0}{2}\int_0^\infty dq\, q J_\nu(qx'_0) J_\nu(qx_0) J_0\left(q\sqrt{t^2-r^2}\right), \tag{3.13}$$

where $\Theta(x)$ is the Heaviside function ($\Theta(x>0) = 1$ and $\Theta(x<0) = 0$). The last integral can also be computed [43] in terms of associated Legendre functions of the first and second kind $(P^\alpha_\beta, Q^\alpha_\beta)$, with the result:

$$G(x_0,x_0',t,r) = \begin{cases} -\dfrac{1}{2\sqrt{2\pi}}\dfrac{1}{\sqrt{\sin(v)}}P^{\frac{1}{2}}_{\nu-\frac{1}{2}}(\cos(v)) & \text{if } \begin{cases}(x_0-x_0')^2+r^2-t^2<0,\\ (x_0+x_0')^2+r^2-t^2>0\end{cases} \\ \\ \dfrac{i}{\sqrt{2}\pi^{\frac{3}{2}}}\dfrac{\sin(\nu\pi)}{\sqrt{\sinh(u)}}Q^{\frac{1}{2}}_{\nu-\frac{1}{2}}(\cosh(u)) & \text{if } \begin{cases}(x_0-x_0')^2+r^2-t^2<0,\\ (x_0+x_0')^2+r^2-t^2<0\end{cases} \\ \\ 0 & \text{if } \quad (x_0-x_0')^2+r^2-t^2>0\,, \end{cases} \tag{3.14}$$

where we defined

$$\begin{aligned} \cos(v) &= 1-\frac{t^2-r^2-(x_0-x_0')^2}{2x_0x_0'}\,, \\ \cosh(u) &= 1+\frac{t^2-r^2-(x_0-x_0')^2}{2x_0x_0'}\,. \end{aligned} \tag{3.15}$$

Note that $v = d(x,x')$ is the timelike distance between x and x' measured in AdS space.

The case when ν is integer is particularly simple since $\sin(\nu\pi)=0$. In that case we obtain

$$G^{\nu\in Z}_{d=2}(x_0,x_0',t,r) = \begin{cases} -\dfrac{\cos(\nu v)}{2\pi\sin(v)} & \text{if} -1<\cos(v)<1, \text{ i.e. } v \text{ is real}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.16}$$

To obtain the propagator in higher dimensions we apply the identity (3.8). Since the propagator depends only on the invariant distance v (as expected) it is better to rewrite that equation as

$$G^\nu_{d+2}(v) = \frac{1}{2\pi\sin(v)}\frac{d}{dv}G^\nu_d(v)\,, \tag{3.17}$$

or

$$G^\nu_{d=2m}(v) = \left(\frac{1}{2\pi\sin(v)}\frac{d}{dv}\right)^{m-1} G^\nu_{d=2}(v)\,. \tag{3.18}$$

with G^ν_2 given by (3.16) for ν integer. While taking the derivatives one must be careful since the condition $-1<\cos(v)<1$ implies a discontinuity for the function at $\cos(v)=\pm 1$, this gives rise to delta functions. To see this better we can rewrite the propagator for $\nu\in Z$ as

$$G^{\nu\in Z}_{d=2m}(v) = -\frac{1}{2\pi}\left(-\frac{1}{2\pi}\frac{d}{d\eta}\right)^{m-1} K_\nu(\eta)\bigg|_{\eta=\cos(v)}\,, \tag{3.19}$$

with

$$K_n(\eta) = \Theta(|\eta|-1)\frac{\cos(n\arccos(\eta))}{\sqrt{1-\eta^2}}$$
$$= \Theta(|\eta|-1)\frac{T_n(\eta)}{\sqrt{1-\eta^2}}, \tag{3.20}$$

where Θ is the Heaviside function and T_n is a Chebyshev polynomial of order n.

In odd dimensional Minkowski space the retarded propagator fills the interior of the forward light cone even when the field is massless. In AdS there is a further subtlety, as seen in fig.1 there is a region in the future of the source where the propagator vanishes. Furthermore, one can show that the filled lightcone touches the boundary along a hyperbola that approaches the (unfilled) lightcone in the boundary of even dimensional Minkowski space as the source approaches the boundary.

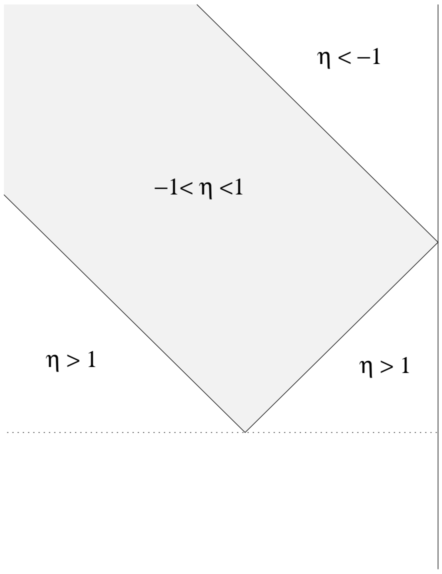

Figure 1: For $d = 2n+1$ and $\nu \in Z$ the propagator is non-vanishing only in the shaded region. Here $\eta = \cos v$, see eq.(3.15).

The calculation proceeds in a similar way in the case of AdS_{3+1} [44]. The expansion in modes reads:

$$G(x_0, x_0', t, r) = -\frac{(x_0 x_0')^{\frac{3}{2}}}{\pi}\int_0^\infty\int_0^\infty dk\, dq\, kq \frac{\sin\left(\sqrt{q^2+k^2}\,t\right)}{\sqrt{q^2+k^2}} J_0(kr) J_\nu(qx_0') J_\nu(qx_0) \tag{3.21}$$

Performing the k integral we obtain [43]

$$G(x_0, x_0', t, r) = -\Theta(t-r)\frac{(x_0x_0')^{\frac{3}{2}}}{2\pi}\int_0^\infty dq\, q J_\nu(qx_0')J_\nu(qx_0)\frac{\cos\left(q\sqrt{t^2-r^2}\right)}{\sqrt{t^2-r^2}} \tag{3.22}$$

This integral is divergent. To handle it we can rewrite it as

$$\frac{(x_0x_0')^{\frac{3}{2}}}{2\pi}\frac{d}{\xi d\xi}\int_0^\infty dq\,\sin(q\xi)J_\nu(qx_0')J_\nu(qx_0)\Big|_{\xi=\sqrt{t^2-r^2}} =$$
$$= \frac{x_0x_0'}{2\pi}\frac{d}{\xi d\xi}\begin{cases} 0 & \text{if } \xi < |x_0'-x_0|\,, \\ \frac{1}{2}P_{\nu-\frac{1}{2}}\left(\frac{x_0^2+{x_0'}^2-\xi^2}{2x_0x_0'}\right) & \text{if } |x_0'-x_0| < \xi < x_0'+x_0\,, \\ -\frac{\cos(\nu\pi)}{\pi}Q_{\nu-\frac{1}{2}}\left(-\frac{x_0^2+{x_0'}^2-\xi^2}{2x_0x_0'}\right) & \text{if } x_0'+x_0 < \xi\,. \end{cases} \tag{3.23}$$

In this case the propagator is simpler for half-integer ν (which includes the case $m=0$). Using the notation of eq.(3.15) the propagator can be written

$$G_{d=3}^{\nu\in Z+\frac{1}{2}}(x_0, x_0', t, r) = \frac{1}{2\pi}\frac{d\tilde{K}_\nu(\eta)}{d\eta}\Big|_{\eta=\cos(v)}\,, \tag{3.24}$$

with

$$\tilde{K}_\nu(\eta) = \Theta(|\eta|-1)\frac{1}{2}P_{\nu-\frac{1}{2}}(\eta)\,. \tag{3.25}$$

Again, using (3.8) we can express the propagator in higher dimensions as:

$$G^{\nu\in Z+\frac{1}{2}}_{d=2m+1}(v) = -\left(-\frac{1}{2\pi}\frac{d}{d\eta}\right)^m \tilde{K}_\nu(\eta)\bigg|_{\eta=\cos(v)}\,. \tag{3.26}$$

Moreover, in the case we are looking at, namely $\nu-\frac{1}{2}$ integer, the function $P_{\nu-\frac{1}{2}}$ is a (Legendre) polynomial of order $\nu-\frac{1}{2}$ and so the derivatives terminate for a given d. This means that the propagator is concentrated on the light-cone, since the only derivatives that remain are due to the jump of the function.

Finally, all the propagators are independent of a choice of a coordinate system. Since they transform as scalars, one only needs to express the invariant distance in the desired coordinates.

4. Examples

4.1 Bubbleography

Let us now use the propagator which was derived in the last section to study the boundary theory description of prototypical probes of the bulk geometry. As a concrete example, we consider a massive source falling along a geodesic in the AdS_{d+1} space. In global coordinates the metric is given by

$$ds^2 = -\cosh^2\mu dt^2 + d\mu^2 + \sinh^2\mu d\Omega^2_{d-1} \tag{4.1}$$

and we will consider a source moving along a geodesic at $\mu'=0$. Primed coordinates will refer to the source, while unprimed coordinates will refer to the point where we evaluate the field. We are interested in the leading behavior of a field that couples to the source close to the boundary where $\mu\to\infty$. To find this we need to integrate the propagator along the worldline of the source, evaluate it in a point a little bit off the boundary and take the limit $\mu\to\infty$ and extract the relevant term. The integral will get contributions from a small segment of the worldline, $0<v<\pi$ and hence we find

$$\phi(\Omega,\mu) = -\frac{1}{2\pi}\int ds\left(\frac{1}{2\pi\sin v}\frac{d}{dv}\right)^{n-1}\frac{\cos\nu v}{\sin v}\,, \tag{4.2}$$

where $n=d/2$. We will only consider integer values of ν. For the $\mu'=0$ worldline $ds=dt'$, and the invariant length is given by

$$\cos v = \cosh\mu\cos(t-t')\,. \tag{4.3}$$

This implies

$$\frac{dt'}{dv} = -\frac{\sin v}{\cosh\mu}\frac{1}{\sqrt{1-\frac{\cos^2 v}{\cosh^2\mu}}} \tag{4.4}$$

and hence

$$\phi(\Omega,\mu) = -\frac{1}{2\pi}\int_0^{\pi} dv \frac{\sin v}{\cosh\mu}\frac{1}{\sqrt{1-\frac{\cos^2 v}{\cosh^2\mu}}}\left(\frac{1}{2\pi\sin v}\frac{d}{dv}\right)^{n-1}\frac{\cos\nu v}{\sin v}\,. \tag{4.5}$$

The integral over v will pick out powers of $\cosh\mu$ such that the leading contribution for $\mu\to\infty$ is

$$\phi = A_{n,\nu}\left(\cosh\mu\right)^{-n-\nu}\,, \tag{4.6}$$

where $A_{n,\nu}$ is a constant that is zero for $\nu+n-1$ odd. Hence we conclude that the boundary field is constant.

In the Poincaré coordinates the situation is different. The worldline is now given by

$$x_0'^2 = 1+t'^2\,, \tag{4.7}$$

implying the line element

$$ds = \frac{dt'}{1+t'^2}\,. \tag{4.8}$$

We find

$$\phi(x,x_0) = -\frac{1}{2\pi}\int_0^{\pi} dv \frac{1}{1+t'^2}\frac{dt'}{dv}\left(\frac{1}{2\pi\sin v}\frac{d}{dv}\right)^{n-1}\frac{\cos\nu v}{\sin v}\,. \tag{4.9}$$

If we use the expression for the worldline in the expression for the invariant length

$$\cos v = 1+\frac{(x_0-x_0')^2+x^2-(t-t')^2}{2x_0'x_0} \tag{4.10}$$

(we have chosen $x'=0$) and define

$$\sin\alpha = \frac{t'}{\sqrt{1+t'^2}}\,, \tag{4.11}$$

$$\sin\beta = \frac{t^2-x_0^2-x^2-1}{\sqrt{4t^2+\left(t^2-x_0^2-x^2-1\right)^2}}\,, \tag{4.12}$$

we obtain

$$\sin(\alpha+\beta) = \frac{-2x_0\cos v}{\sqrt{4t^2+\left(t^2-x_0^2-x^2-1\right)^2}}\,. \tag{4.13}$$

We can now obtain (noting that β is independent of v)

$$\phi(x,x_0) = -\frac{1}{2\pi}\int_0^{\pi} dv\frac{d\alpha}{dv}\left(\frac{1}{2\pi\sin v}\frac{d}{dv}\right)^{n-1}\frac{\cos\nu v}{\sin v} \tag{4.14}$$

$$= -\int_0^{\pi} dv A\sin v\frac{1}{\sqrt{1-A^2\cos^2 v}}\left(\frac{1}{2\pi\sin v}\frac{d}{dv}\right)^{n-1}\frac{\cos\nu v}{\sin v}\,, \tag{4.15}$$

where

$$A = \frac{-2x_0}{\sqrt{4t^2+\left(t^2-x_0^2-x^2-1\right)^2}}\,. \tag{4.16}$$

The v integration results in

$$\phi = B_{n,\nu} \frac{x_0^{n+\nu}}{\left(4t^2 + (t^2 - x_0^2 - x^2 - 1)^2\right)^{\frac{n+\nu}{2}}}, \tag{4.17}$$

which implies a boundary field expectation value of the form

$$\langle \mathcal{O} \rangle = C_{n,\nu} \frac{1}{\left(4t^2 + (t^2 - x_0^2 - x^2 - 1)^2\right)^{\frac{n+\nu}{2}}}, \tag{4.18}$$

where $B_{n,\nu}$ and $C_{n,\nu}$are constants that are zero for $\nu + n - 1$ odd. As an example of the scale/distance duality between the boundary and the bulk, in [22] it was shown that an object at a fixed distance from a horizon would look like an extended blob in the boundary theory; the closer to the horizon the larger it will appear. A freely falling object is then expected to grow as it closes in on the horizon. The above calculation shows that this is not the whole story. Even if the boundary description of the object starts out like a blob it will develop into an expanding bubble over time as illustrated in fig.2. This can be understood intuitively in the following way. Since the object is falling, all scales will expand but at different rates. The points in the center of the bubble depend on the source at later times, when the information has not had time to spread much in the transverse directions. On the other hand, the source is falling faster at later times and therefore corresponds to a region in the boundary that must expand faster. The result is a shockwave that produces a bubble.

In the case of AdS_3 we can also verify that the change of coordinates from global to Poincaré amounts to a conformal transformation in the boundary; the bubble is hence a conformal transformation of the constant profile. In fact

$$4t^2 + \left(t^2 - x^2 - 1\right)^2 = (1 + z^2)(1 + \overline{z}^2), \tag{4.19}$$

(with z as in the previous section) and since

$$\frac{\partial w}{\partial z} = \frac{2}{1 + z^2}, \tag{4.20}$$

we find that the bubble is simply given by the conformal scaling prefactors

$$\langle \mathcal{O} \rangle = C_{n,\nu} \left(\frac{1}{4} \frac{\partial w}{\partial z} \frac{\partial \overline{w}}{\partial \overline{z}} \right)^{\frac{n+\nu}{2}}. \tag{4.21}$$

As we see, in the case of AdS_3, a coordinate transformation in the bulk induces a conformal transformation in the boundary and the boundary field transforms accordingly. The geodesic $\mu' = 0$ where the particle is always at the center in global coordinates was particularly simple. However any other geodesic can be obtained from that one using an isometry of AdS_3 space and so the boundary field produced by such particle can be obtained by the corresponding conformal transformation in the boundary.

In Poincaré coordinates the trajectory is $x_0'^2 = 1 + t'^2$. Performing the isometry $x_0 \to x_0/a$, $t \to t/a$,$x \to x/a$, with a an arbitrary parameter we obtain the new trajectory defined by

$$x_0'^2 = a^2 + t'^2 \,. \tag{4.22}$$

In the boundary, the conformal transformation induced by these coordinates is simply a rescaling of $z, \bar{z}$. However in global coordinates the conformal transformation is not so trivial and is given by

$$\tan\left(\frac{w'}{2}\right) = a \tan\left(\frac{w}{2}\right), \quad \tan\left(\frac{\bar{w}'}{2}\right) = a \tan\left(\frac{\bar{w}}{2}\right), \tag{4.23}$$

where w' are the new coordinates. Since in the coordinates w the boundary field is a constant we obtain the field produced by the particle following the new geodesic as

$$\begin{aligned} \phi(w', \bar{w}') &= \left(\frac{dw}{dw'}\right)^{\frac{\nu+1}{2}} \left(\frac{d\bar{w}}{d\bar{w}'}\right)^{\frac{\nu+1}{2}} &(4.24) \\ &= \frac{a^{\nu+1}}{\left(a^2 \cos^2 \frac{w'}{2} + \sin^2 \frac{w'}{2}\right)^{\frac{\nu+1}{2}} \left(a^2 \cos^2 \frac{\bar{w}'}{2} + \sin^2 \frac{\bar{w}'}{2}\right)^{\frac{\nu+1}{2}}} \,. &(4.25) \end{aligned}$$

As a check it can be seen that the same expression is obtained if one plugs in the new geodesic in the previous formulas with the propagator. The resulting field looks for small times also like an expanding bubble but is of very different nature since it is obviously periodic in time.

A more interesting case is that of BTZ coordinates. The conformal transformation from the coordinates w where the field is constant to BTZ coordinates reads:

$$\tanh\left((r_+ - r_-)\frac{\phi_+}{2}\right) = a \tan\left(\frac{w}{2}\right), \qquad \tanh\left((r_+ + r_-)\frac{\phi_-}{2}\right) = a \tan\left(\frac{\bar{w}}{2}\right) \tag{4.26}$$

and the field is accordingly

$$\phi(\phi_+, \phi_-) = \frac{a^{\nu+1}(r_+^2 - r_-^2)^{\frac{\nu+1}{2}}}{\left(a^2 + (1+a^2)\sinh^2 \frac{(r_+ - r_-)\phi_+}{2}\right)^{\frac{\nu+1}{2}} \left(a^2 + (1+a^2)\sinh^2 \frac{(r_+ - r_-)\phi_-}{2}\right)^{\frac{\nu+1}{2}}} \,. \tag{4.27}$$

In the black hole case, we must perform the identification $\phi = \phi + 2\pi$ which amounts to a sum over images:

$$\phi_{\mathrm{BTZ}}(\phi_+, \phi_-) = \sum_{n=-\infty}^{\infty} \phi(\phi_+ + 2\pi n, \phi_- + 2\pi n) \,. \tag{4.28}$$

The field is depicted in fig.2 where $\phi = -\pi \ldots \pi$.

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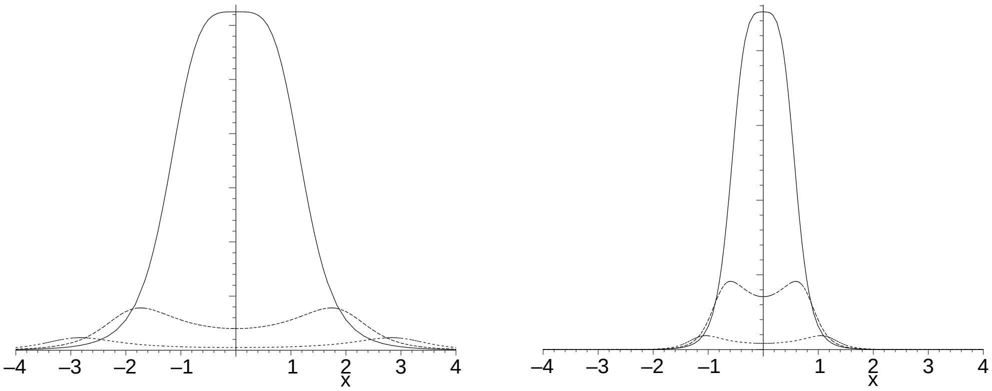

Figure 2: Dilaton field in the boundary of, left: AdS_3 for $\nu = 2$ and times $t = 1, 2, 3$ and right: a BTZ black hole for $\nu = 2$, $r_+ = 1t = 0.5, 0.8, 1.2$.

4.2 Supersymmetric YM at zero temperature

From equation (4.6) it follows that the above picture is no longer relevant if $\nu - n + 1$ is odd. In these cases the expectation value in the boundary theory vanishes. In particular this is true for massless dilatons, which in the case of AdS_5 implies that the expectation value $< F^2 >$ must vanish for a bulk source in a free fall. How can one then understand the nonvanishing expectation value obtained in [22]?

The important point is that in [22] the source is held at a fixed position in Poincaré coordinates. This can be achieved by suspending it on a string hanging down from the boundary with an appropriate tension. The other endpoint of the string will then show up as a point charge in the boundary theory. We conclude that nonvanishing expectation values of F^2 must be supported by charges. To be more precise we can use the propagator derived above to calculate $\langle F^2 \rangle$ due to a point charge by representing the charge by a string that goes straight down to the horizon. We must then consider the string as an extended source for the dilaton field and integrate the propagator along the string.

To perform our calculations we must know how F^2 couples to the supergravity fields. There are two such fields that will be of interest to us, the dilaton ϕ, and the volume. We will be interested in variations with respect to certain linear combinations of these fields and a convenient basis is to consider variations either with the string metric or the Einstein metric held fixed. To fix the conventions, let us consider the D-brane action

$$I_{BI} = Tr\mathcal{T}_p \int d^{p+1}x e^{-\phi} \sqrt{\det(g_{\mu\nu} + \mathcal{F}_{\mu\nu} + g^{ij}\partial_\mu X_i \partial_\nu X_j)} \,. \tag{4.29}$$

Following e.g. [45, 18] we then rescale the worldvolume gauge field strength,

$$\mathcal{F} = \mathcal{T} F \,, \tag{4.30}$$

where $\mathcal{T}$ is the string tension,

$$\mathcal{T} = \frac{1}{2\pi\alpha'} \tag{4.31}$$

and substitute the Dp-brane tension

$$\mathcal{T}_p = g_s^{-1}(2\pi)^{\frac{1-p}{2}}\mathcal{T}^{\frac{p+1}{2}}\,. \tag{4.32}$$

Expanding to second order in F we find

$$I_{BI} = Tr\int d^{p+1}x\sqrt{-g}\,e^{-\phi}\left(\mathcal{T}_p + \frac{1}{4g_{YM}^2}\left(g^{\mu\nu}F_{\mu\nu}\right)^2 + \cdots\right), \tag{4.33}$$

where the Yang-Mills coupling is given by

$$g_{YM}^2 = (2\pi)^{p-2}g_s\alpha'^{\frac{p-3}{2}}\,. \tag{4.34}$$

It is convenient to define a new metric $\widehat{g}_{\mu\nu}$, defined through

$$g_{\mu\nu} = e^{\frac{2}{p+1}\phi}\widehat{g}_{\mu\nu}\,, \tag{4.35}$$

which for $p = 3$ coincides with the Einstein metric. The Born-Infeld action then becomes

$$I_{BI} = Tr\int d^{p+1}x\sqrt{-\widehat{g}}\left(\mathcal{T}_p + \frac{1}{4g_{YM}^2}e^{-\frac{4}{p+1}\phi}\left(\widehat{g}^{\mu\nu}F_{\mu\nu}\right)^2 + \cdots\right). \tag{4.36}$$

If we furthermore define

$$\widehat{\phi} = \frac{4}{p+1}\phi\,, \tag{4.37}$$

we find that F^2 is obtained as

$$\frac{\delta I_{BI}}{\delta\widehat{\phi}} = -\frac{1}{4g_{YM}^2}F^2\,. \tag{4.38}$$

For $p = 3$ and $p = 4$, we have checked that $\widehat{\phi}$ fluctuations keeping $\widehat{g}_{\mu\nu}$ fixed are massless. On the supergravity side, the supergravity action (dimensionally reduced over the unit sphere) is then given by

$$S = -\frac{\Omega_{8-p}}{4\kappa_{10}^2}\int d^{p+2}x\sqrt{-g_E}\,g_E^{\mu\nu}\partial_\mu\widehat{\phi}\partial_\nu\widehat{\phi}\,, \tag{4.39}$$

where Ω_{8-p} is the volume of the unit $8-p$ sphere. If we have a string as a source we will need to vary

$$S_{int} = \frac{1}{2\pi\alpha'}\int d^2x\sqrt{-g} = \frac{1}{2\pi\alpha'}\int d^2x e^{\widehat{\phi}/2}\sqrt{-\widehat{g}}\,, \tag{4.40}$$

where the metrics are now the induced ones on the worldsheet. We see that the variation with respect to $\widehat{\phi}$ gives

$$\delta S_{int} = \frac{1}{4\pi\alpha'}\int d^2x\sqrt{-g}\,\delta\widehat{\phi}\,. \tag{4.41}$$

The corresponding variation of the supergravity action is then given by the boundary term

$$\delta S = \frac{\Omega_{8-p}}{2\kappa_{10}^2}\int d^{p+1}x\sqrt{-g}g^{UU}e^{-2\bar{\phi}}\partial_U\widehat{\phi}\delta\widehat{\phi}\Big|_{U\to\infty}\,, \tag{4.42}$$

where $\bar{\phi}$ is the background value of the dilaton. Hence it follows that

$$\frac{1}{4g_{YM}^2}\left\langle F^2\right\rangle = \frac{\Omega_{8-p}}{2\kappa_{10}^2}\sqrt{-g}g^{UU}e^{-2\bar{\phi}}\partial_U\widehat{\phi}\Big|_{U\to\infty}\,, \tag{4.43}$$

if we consider string like sources.

Let us apply this to the case of a string hanging straight down when $p=3$, i.e, $d=4$. The string traces out a worldsheet that gives rise to a ϕ field

$$\phi(U,x) = -\frac{1}{4\pi\alpha'}\frac{2\kappa_{10}}{\Omega_5}\int dt'dU'\sqrt{-g_2}\frac{1}{2\pi\sin v}\frac{d}{dv}\frac{\cos 2v}{\sin v}\,. \tag{4.44}$$

If we want to write the AdS_5 metric as

$$ds^2 = \alpha'\left(\left(\frac{R}{U}\right)^2 dU^2 + \left(\frac{U}{R}\right)^2\left(\sum_{i=1}^{3}dx_i^2 - dt^2\right) + R^2 d\Omega_5^2\right), \tag{4.45}$$

where $R^2 = g_{YM}\sqrt{N}$ we need to take $x \to x/R^2$, $t \to t/R^2$, rescale the metric by $\alpha' R^2$ and put $x_0 = 1/U$. Noting that $\delta^{(3)}(x) = \frac{1}{R^6}\delta^{(3)}(x/R^2)$ we find

$$\phi(U,x) = -\frac{1}{8\pi^2R^6}\frac{2\kappa_{10}}{\Omega_5}\int dvdU'\frac{U^{-1}U'^{-1}}{\sqrt{U^{-2}+U'^{-2}+\frac{r^2}{R^4}-2U^{-1}U'^{-1}\cos v}}\frac{d}{dv}\frac{\cos 2v}{\sin v}\,, \tag{4.46}$$

where we have put $r^2 = \sum_{i=1}^{3}x_i^2$ and also used

$$\frac{t-t'}{R^2} = \sqrt{U^{-2}+U'^{-2}+\frac{r^2}{R^4}-2U^{-1}U'^{-1}\cos v}\,. \tag{4.47}$$

We find

$$\phi(U,x) = -\frac{15}{256\pi^2R^6}\frac{2\kappa_{10}}{\Omega_5}\int dU'\frac{U^{-4}U'^{-4}}{\left(U'^{-2}+\frac{r^2}{R^4}\right)^{7/2}} = -\frac{1}{128\pi^2}R^2U^{-4}\frac{1}{r^4}\,, \tag{4.48}$$

hence

$$\frac{1}{4g_{YM}^2}F^2 = \frac{1}{32\pi^2}g_{YM}\sqrt{N}\frac{1}{r^4}\,, \tag{4.49}$$

which suggests that the effective charge goes like $\sqrt{g_{YM}\sqrt{N}}$. Hence the force on another charge $\sqrt{g_{YM}\sqrt{N}}$ is $\sim g_{YM}\sqrt{N}\frac{1}{r^2}$ implying a potential $\sim g_{YM}\sqrt{N}\frac{1}{r}$, in agreement with [7].

4.3 Supersymmetric YM at nonzero temperature

We may also consider the nonzero temperature case. The starting point is the metric of a non-extremal p-brane, given by

$$\frac{ds^2}{\alpha'} = \left(\frac{R}{U}\right)^{\frac{7-p}{2}} \frac{dU^2}{f(U)} - \left(\frac{U}{R}\right)^{\frac{7-p}{2}} f(U)dt^2 + \left(\frac{U}{R}\right)^{\frac{7-p}{2}} \sum_{i=1}^{p} dx_i^2 + R^{\frac{7-p}{2}} U^{\frac{p-3}{2}} d\Omega_{8-p}^2 \,,$$
$$f(U) = 1 - \frac{U_T^{7-p}}{U^{7-p}} \,, \tag{4.50}$$

while the dilaton is given by

$$e^{\phi} = g_s \alpha'^{\frac{p-3}{2}} \left(\frac{R^{7-p}}{U^{7-p}}\right)^{\frac{3-p}{4}} \,. \tag{4.51}$$

We have defined

$$R^{\frac{7-p}{2}} = g_{YM}\sqrt{d_p N} \,, \tag{4.52}$$

where

$$d_p = 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma(\frac{7-p}{2}) \tag{4.53}$$

and g_{YM} is given as before.

We will limit ourselves to the case of $d = 4$ and begin by deriving the propagator for a static configuration since we can no longer use the exact retarded propagator evaluated in the previous section.

The metric (4.50) reduces to

$$ds^2 = \frac{R^2}{U^2 f(U)} dU^2 - \frac{U^2 f(U)}{R^2} dt^2 + \frac{U^2}{R^2} dx_i^2 + R^2 d\Omega_5^2 \,, \tag{4.54}$$
$$f(U) = 1 - \left(\frac{U_T}{U}\right)^4 \,. \tag{4.55}$$

In this case we will consider only the case of a static source coupled to a massless field ϕ through the action

$$S_{\text{int}} = \int ds \phi(x(t)) = \int dt \sqrt{|g_{tt}|} \phi(x(t)) \,. \tag{4.56}$$

In the propagator equation (3.1), this amounts to multiplying the right hand side by $\sqrt{|g_{tt}|}$ and omitting the time derivatives. The resulting equation is

$$\frac{1}{U^3}\partial_U \left(\frac{U^5 f(U)}{R^2} \partial_U G(U,U',x_i)\right) + \frac{R^2}{U^2}\partial_{ii} G(U,U',x_i) = \frac{\sqrt{f(U)}}{R^3 U^2} \delta(U-U')\delta^{(3)}(x_i) \,. \tag{4.57}$$

The boundary conditions are that G is regular at $U = U_T$ and that G vanishes at $U \to \infty$. Rescaling the coordinates as $U \to U_T u$, $t \to tR^2/U_T$ and $x \to xR^2/U_T$, the equation simplifies to

$$(u^5 - u)\partial_{uu} G + (5u^4 - 1)\partial_u G + u\partial_{ii} G = \frac{\sqrt{u^4-1}}{R^7 u} \delta(u-u')\delta^{(3)}(x_i) \,. \tag{4.58}$$

Fourier transforming both sides of eq.(4.57) with respect to x_i leads to

$$(u^5-u)\partial_{uu}\tilde{G}+(5u^4-1)\partial_u\tilde{G}-uk^2\tilde{G}=\frac{\sqrt{u^4-1}}{R^7u}\delta(u-u')\,, \tag{4.59}$$

with

$$\tilde{G}(u,u',k)=\int d^3ke^{i\vec{k}\vec{x}}G(u,u',x_i)\,. \tag{4.60}$$

For $u<u'$ and for $u>u'$ the Green function is given by (different) solutions of the homogeneous equation

$$(u^5-u)y''(u)+(5u^4-1)y'(u)-uk^2y(u)=0\,. \tag{4.61}$$

The homogenous equation has been obtained in [17] and discussed at length in [46]. For $u>u'$ there are two solutions which we shall call $y_1(u,k)$ and $\tilde{y}_1(u,k)$ which satisfy

$$\begin{aligned} y_1(u,k) &= \frac{1}{u^4}+\mathcal{O}\left(\frac{1}{u^5}\right), && u\to\infty\,,\\ \tilde{y}_1(u,k) &= 1+\mathcal{O}\left(\frac{1}{u},\frac{\ln(u)}{u^4}\right), && u\to\infty\,, \end{aligned} \tag{4.62}$$

and for $u<u'$ there is another pair of solutions $y_2(u,k)$, $\tilde{y}_2(u,k)$ satisfying

$$\begin{aligned} y_2(u,k) &= 1+\mathcal{O}(u-1)\,, && u\to 1\,,\\ \tilde{y}_2(u,k) &= \ln(u-1)+\mathcal{O}(u-1,(u-1)\ln(u-1))\,, && u\to 1\,. \end{aligned} \tag{4.63}$$

The boundary conditions then require that

$$\tilde{G}(u>u',k)=Ay_1(u,k)\,,\qquad \tilde{G}(u<u',k)=By_2(u,k)\,, \tag{4.64}$$

and A and B should be chosen such that

$$By_2(u',k)-Ay_1(u',k) = 0\,, \tag{4.65}$$

$$By_2'(u',k)-Ay_1'(u',k) = \frac{1}{R^7u^2\sqrt{u^4-1}}\,. \tag{4.66}$$

It follows that

$$A=\frac{y_1}{y_1y_2'-y_2y_1'}\frac{1}{R^7u^2\sqrt{u^4-1}}\,, \tag{4.67}$$

$$B=\frac{y_2}{y_1y_2'-y_2y_1'}\frac{1}{R^7u^2\sqrt{u^4-1}}\,, \tag{4.68}$$

where the functions are evaluated at $u=u'$. As follows from (4.61), the Wronskian $W(y_1,y_2)=y_1y_2'-y_2y_1'$ is given by

$$W(y_1,y_2)=\frac{w(k)}{u^5-u}\,, \tag{4.69}$$

where $w(k)$ is a function that depends only on k. This simplifies the expression for the propagator which turns out to be

$$\tilde{G}(u,u',k) = \frac{1}{w(k)} y_1(u_>,k) y_2(u_<,k) \frac{\sqrt{u'^4-1}}{u'R^7}\,, \tag{4.70}$$

with $u_< = \min(u,u')$ and $u_> = \max(u,u')$. In ref. [46] a series expansion was obtained for y_1 and y_2 and also for $w(k)$. There it was shown that for real k there are no zeros of w (and so the propagator is well-defined) and also that there is an infinite number of zeros on the imaginary axes. These zeros give precisely the masses of the glueballs in [17].

For what follows, besides the expansion of [46] it will be useful to have also an approximate but simpler expression for $w(k)$ which we derive in the appendix using the WKB method.

A charge in the boundary is represented in the bulk by a string hanging from the boundary down to the horizon. The minimal action configuration corresponds to the string extended only in the U and t directions. Such a string will be a source for the dilaton through the coupling

$$S_{\text{int}} = \frac{1}{4\pi} \int dU dt \sqrt{g_{tt} g_{UU}} \phi = \frac{1}{4\pi} \int du dt \sqrt{g_{tt}} \frac{uR}{\sqrt{u^4-1}} \phi\,, \tag{4.71}$$

where we have put $\alpha' = 1$ for simplicity. The boundary value of ϕ corresponds to the expectation value of F^2 in the presence of the charge. We know that for short distances it will behave as $1/r^4$ since in that region we can replace the black hole backround by AdS space and so (4.49) is reproduced. For large distances it should decay exponentially as $\exp(-r/L_S)$ where L_S is the screening length. The value of the dilaton can be computed using the static Green function as

$$\phi(u,x_i) = \frac{1}{4\pi} \int du' \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \frac{uR}{\sqrt{u^4-1}} \tilde{G}(u,u',k)\,. \tag{4.72}$$

Using (4.70) the value near the boundary is given by

$$\phi(U,x) \approx_{U\to\infty} \frac{1}{U^4} \phi_0(x)\,, \tag{4.73}$$

$$\phi_0(x) = \frac{U_T^4}{4\pi R^6} \int du' \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \frac{y_2(u',k)}{w(k)}\,. \tag{4.74}$$

The angular integral in the Fourier transform can be performed with the result

$$\phi_0(x) = \frac{U_T^4}{(2\pi)^3 R^6} \int du' \int_0^\infty dk \frac{\sin(kr)}{kr} \frac{y_2(u',k)}{w(k)}\,, \tag{4.75}$$

where $r^2 + x_1^2 + x_2^2 + x_3^2$. The integral in k can be extended from $-\infty \to \infty$ and evaluated by the method of residues. The integrand has simple poles in the zeros of $w(k)$ which are located on the imaginary axis. A zero in $w(k)$ means that the solutions y_1 and y_2 are proportional, that is that for that value of k there exists a solution well behaved in $u = 1$ and $u \to \infty$. Such solutions are the eigenfunctions computed in [17, 46] and so, the zeros of $w(m_n) = 0$ define the glueball masses in the $2 + 1$ confining theory.

$$\phi_0(x) = \frac{U_T^4}{8\pi R^6} \sum_{n=1}^{\infty} \frac{e^{-m_n r}}{m_n r w'(m_n)} \int du' y_2(u', m_n) du' \,, \tag{4.76}$$

where m_n and $y_2(u', m_n)$ are the n eigenvalue and eigenfunction of eq.(4.61). Note that the integral in u' is well defined for the eigenfunctions. Approximate values of the eigenvalues and eigenfunctions are obtained in the Appendix.

At long distances the function decays exponentially with a screening length given by

$$L_S = \frac{R^2}{U_T m_{gl.}} \,, \tag{4.77}$$

where $m_{gl} \approx 3.4$ is the lowest eigenvalue. From the boundary point of view this was expected, since at large temperatures the theory is reduced to one less dimension. The mass gap in this dimensionally reduced theory is $m_{gl.}$ which in the original theory is interpreted as a screening mass.

Hence we find that the screening length is determined by the glueball masses and is of order $1/T = R^2/U_T$.

4.4 Confinement and flux tubes in non supersymmetric YM

The metric (4.50) can also be used to describe nonsupersymmetric QCD$_p$, [13]. To achieve this in the metric (4.50) we must go first to Euclidean time $t \to \tau$ and then go back to Minkowski signature but through $x_p \to t$. To avoid a conical singularity at $U = U_T$, τ must be periodic with period $\pi/T = \pi R^2/U_T$. As a result the boundary theory is a SYM theory compactified on a circle and with coupling constant $g_{YM}^2 T$, where g_{YM}^2 is the coupling constant of the higher dimensional theory.

The potential between two quarks in the boundary is described by the minimal action of a string going from one quark to another and extending in the bulk. When the distance between the two charges is very large, the string hangs down to $U = U_T$ where the metric is approximately flat and the energy is proportional to the separation length giving confinement. This string is a source for the dilaton and so in the boundary is seen as a non vanishing $\langle F^2 \rangle$ which will be concentrated along a flux tube connecting the two quarks.

The tension of the flux tube was calculated in [8] using the picture of a hanging string with the result

$$\sigma_p = \frac{1}{2\pi} \left(\frac{U_T}{R} \right)^{\frac{7-p}{2}} \,. \tag{4.78}$$

This can be obtained also in a different way. The free energy $A(\beta,\lambda)$ and the gluon condensate are related by

$$-\lambda\frac{\partial_\lambda A}{\partial\lambda} = \frac{1}{4\lambda}\int d^dx\langle F^2\rangle\,, \tag{4.79}$$

where we defined $\lambda = g_{YM}^2$. In this case the gluon condensate is known from the boundary value of the dilaton and so the free energy can be obtained up to a constant as

$$A = -\int\frac{d\lambda}{\lambda}\int d^dx\frac{1}{4\lambda}\langle F^2\rangle\,. \tag{4.80}$$

To proceed, we first need the dilaton equation

$$\frac{1}{2\kappa_{10}^2}\partial_\mu\sqrt{-g}\,g^{\mu\nu}e^{-2\bar\phi}\partial_\nu\widehat{\phi} = \frac{1}{4\pi\alpha'}\sqrt{-g_2}\delta\,. \tag{4.81}$$

Let us integrate over the $8-p$ dimensions of the internal sphere and the $p-2$ transverse directions to the flux tube, i.e. all directions transverse to the string except the radial U and the angular τ. We put $\chi = \int d^6x\phi$ and obtain

$$\frac{1}{2\kappa_{10}^2}\partial_U\sqrt{-g}g^{UU}e^{-2\phi}\partial_U\chi = \frac{1}{4\pi}\left(\frac{U}{R}\right)^{\frac{7-p}{2}}\delta^{(2)}\left(U-U_T\right). \tag{4.82}$$

Integrating over the (U,τ) plane then gives rise to a boundary term at $U\to\infty$. Note that the origin of the radial U coordinate at $U=U_T$ (the horizon of the Euclidean black hole) is a regular point since the period T has been chosen in an appropriate way. We find

$$\frac{\Omega_{8-p}}{2\kappa_{10}^2}\int d^{p-2}xd\tau\sqrt{-g}g^{UU}e^{-2\bar\phi}\partial_U\widehat{\phi} = \frac{1}{4\pi}\left(\frac{U_T}{R}\right)^{\frac{7-p}{2}}. \tag{4.83}$$

Note that the leading contribution to ϕ is constant over the S^{8-p} near the boundary. (In the case p=3 one has that the Kaluza-Klein states on the sphere correspond to states with higher conformal dimension). Using the previous results we finally find that the integrated gluon condensate is given by

$$\int d^{p-2}xd\tau\frac{1}{4g_{YM}^2}F^2 = \frac{1}{4g_{YM}^2T}\int d^{p-2}xF^2 = \frac{1}{4\pi}\left(\frac{U_T}{R}\right)^{\frac{7-p}{2}}. \tag{4.84}$$

We must now relate the gluon condensate to the tension of the flux tube. In the perturbative case the gluon condensate is proportional to λ and the energy and gluon condensate are equal. The gluon condensate calculated above is, however, given by $\frac{1}{4\lambda}\langle F^2\rangle\sim\sqrt{\lambda}$. The reasoning is identically the same as in the case of (4.40). The result is a free energy that is twice as large as the gluon condensate[7]. This implies

[7]For a D-string the factor is -2.

that (4.84) correctly reproduces the tension (4.78). More generally we see that, for massless fields, the total energy in the boundary theory is obtained by integrating the source over the bulk.

In fact $\langle F^2 \rangle$ can also be computed as a function of the position and that gives the profile of the string. We first will look at the case of a $2+1$ dimensional theory and then at the $3+1$ case.

When $p = 3$, the metric is given by

$$ds^2 = \frac{R^2}{U^2 f(U)} dU^2 + \frac{U^2 f(U)}{R^2} d\tau^2 + \frac{U^2}{R^2}(dx_1^2 + dx_2^2 - dt^2) + R^2 d\Omega_5^2\,, \quad (4.85)$$

$$f(U) = 1 - \left(\frac{U_T}{U}\right)^4, \quad (4.86)$$

where we have let x_3 become our new time coordinate t. A difference with respect to the non-confining case is that at $U = U_T$ there are geodesics of constant U. In fact in that region the metric is approximately a product metric of an R^{2+1} parameterized by $x_{1,2}, t$, a five sphere and a plane parameterized by U, τ with $\sqrt{U - U_T}$ a radial coordinate and τ an angle. From now on we will consider fields that are independent of the coordinates of the sphere.

A particle sitting at the center $U = U_T$ of the (Euclidean) black hole will remain at rest. Furthermore, a freely falling source will eventually be trapped around the R^{2+1} at the center and henceforth represent a stable configuration. This is the reason for the existence of a mass gap and stable glue balls as well as of confinement. If the particle is a source for the dilaton, then it will induce in the boundary an expectation value of $\langle F^2 \rangle$ which will look like a stable blob and be interpreted by the boundary observer as some kind of a glueball state. Consider now the more physical case of a string sitting at the center of the (euclidean) black hole and extended for example in the x_1 direction. In the boundary it will give rise to a flux tube with a shape that we proceed to calculate.

First the dilaton field ϕ produced by the string should be obtained. The field is independent of t and x_1 since the source is static and extended in x_1 and is also independent of τ since the string is at the center of the $\rho = \sqrt{U - U_T}, \tau$ plane. The equation (4.81) for ϕ is:

$$\partial_U \Big(U^5 f(U) \partial_U \phi(U, x_2)\Big) + R^4 U \partial_{22} \phi(U, x_2) = \frac{2\kappa_{10}^2}{4\pi} \frac{U_T^2}{R^2 \Omega_5} \delta(U - U_T) \delta(x_2)\,, \quad (4.87)$$

where Ω_5 is the volume of the unit five-sphere. Rescaling as before $U \to U_T u$, $t \to t R^2/U_T$ and $x \to x R^2/U_T$, the equation simplifies to

$$(u^5 - u)\partial_{uu}\phi + (5u^4 - 1)\partial_u \phi - u k^2 \phi = \frac{2\kappa_{10}^2}{4\pi} \frac{1}{U_T R^2 \Omega_5} \delta(u - 1)\,, \quad (4.88)$$

where the Fourier transform

$$\phi(u, k) = \int dk e^{ik} x_2 \phi(u, x_2) \quad (4.89)$$

was introduced. The homogeneous equation is the same as (4.61). In terms of its solution the field can be written as

$$\phi(u,k) = A y_1(u,k)\,. \tag{4.90}$$

The solution $y_1(u,k)$ behaves as $\ln(u-1)$ when $u \to 1$ as is needed to match the delta function on the right hand side. In fact we need

$$\phi(u,k) \approx_{u\to 1} \frac{1}{4}\frac{2\kappa_{10}^2}{4\pi}\frac{1}{U_T R^2 \Omega_5}\ln(u-1) \tag{4.91}$$

On the other hand we have

$$\begin{aligned} y_1(u,k) &= \alpha y_2(u,k) + \beta \tilde{y}_2(u,k) &(4.92)\\ \beta &= -\frac{y_1 y_2' - y_2 y_1'}{y_2 \tilde{y}_2' - y_2' \tilde{y}_2} = -\frac{W(y_1,y_2)}{W(y_2,\tilde{y}_2)} = -\frac{w(k)}{4}\,, &(4.93)\end{aligned}$$

where $w(k)$ is the function defined in (4.69).

The constant A can now be obtained since

$$\begin{aligned} \phi(u,k) &= A y_1(u,k) = \alpha y_2(u,k) + \beta \tilde{y}_2(u,k) \approx_{u\to 1} A\beta \ln(u-1) &(4.94)\\ \Rightarrow A &= -\frac{2\kappa_{10}^2}{4\pi}\frac{1}{U_T R^2 \Omega_5}\frac{1}{w(k)}\,. &(4.95)\end{aligned}$$

The field created by the static source is then given by

$$\phi(u,x_2) = -\frac{2\kappa_{10}^2}{4\pi}\frac{1}{U_T R^2 \Omega_5}\int \frac{dk}{(2\pi)} e^{-ikx}\frac{1}{w(k)} y_1(u,k)\,. \tag{4.96}$$

Using that near the boundary $y_1 \approx u^{-4} = U_T^4/U^4$ and using the relation (4.43) between the boundary field and the gluon condensate, it follows that

$$\frac{1}{4g_{YM}^2}\langle F^2\rangle = \frac{4U_T^3}{4\pi R^4}\int \frac{dk}{(2\pi)} e^{-ikx_2}\frac{1}{w(k)}\,. \tag{4.97}$$

This will be seen by the boundary observer as an infinite string of finite width. In the case of two charges separated by a large distance the string will join both charges. As a check, the integral of this profile over the transverse direction x_2 is given by the value of the Fourier transform at $k=0$. Since $w(0)=4$ (as can be deduced solving the differential equation exactly for $k=0$) the result (4.84) is reproduced.

The profile of the QCD string is easily seen to decay exponentially at large distance as $\exp(-m_{\mathrm{gl.}} x_2)$ with $m_{\mathrm{gl.}}$ the lowest glueball mass. An approximation to the profile can be obtained using the function $w(k)$ computed in the Appendix and is

$$\begin{aligned} \frac{1}{4g_{YM}^2}\langle F^2\rangle &\approx \frac{4U_T^3}{4\pi R^4}\int \frac{dk}{(2\pi)}\frac{\pi\left(k^2+\left(\frac{\pi}{2\alpha}\right)^2\right)}{16\sqrt{2}}\frac{1}{\cosh(\alpha k)} e^{-ikx_2}\\ &= \frac{\pi^2}{2^{\frac{15}{2}}}\frac{U_T^3}{R^4\alpha^3}\frac{1}{\cosh^3\left(\frac{U_T x_2}{2\alpha R^2}\right)}\,, &(4.98)\end{aligned}$$

where in the last line we return to the original variable $x_2 \to U_T x_2/R^2$ and α is the constant defined in the Appendix as

$$\alpha = \frac{1}{4\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 . \tag{4.99}$$

This profile is depicted in fig.(3).

For the $p = 4$ case the metric and expectation value of the dilaton are given by

$$\begin{aligned} ds^2 &= \left(\frac{R}{U}\right)^{\frac{3}{2}}\frac{dU^2}{f(U)} + \left(\frac{U}{R}\right)^{\frac{3}{2}} f(U)d\tau^2 + \left(\frac{U}{R}\right)^{\frac{3}{2}}\left(\sum_{i=1}^{3} dx_i^2 - dt^2\right) + R^{\frac{3}{2}}U^{\frac{1}{2}}d\Omega_4^2 \,, \\ e^{\bar{\phi}} &= g_s\left(\frac{U}{R}\right)^{\frac{3}{4}} , \\ f(U) &= 1 - \frac{U_T^3}{U^3} \,. \end{aligned} \tag{4.100}$$

Considering a string situated at $U = U_T$ and extending along x_3,t as a source, the equation for the massless scalar $\widehat{\phi}$ can then be written as:

$$\partial_U\left(U^4\left(1-\frac{U_T^3}{U^3}\right)\partial_U\widehat{\phi}\right) + UR^3\partial_{ii}\widehat{\phi} = \frac{g_s^2 R^{\frac{3}{2}}}{4\pi\Omega_4}2\kappa_{10}^2\sqrt{-g_{tt}g_{33}}\delta^{(2)}(x_{1,2})\delta(U-U_T)\,, \tag{4.101}$$

where Ω_4 is the volume of the 4-sphere. Rescaling the variables as $U = U_T u$ and $x_i \to x_i R^{\frac{3}{2}}/\sqrt{U_T}$ the equation reduces to

$$(u^4-u)\partial_{uu}\widehat{\phi} + (4u^3-1)\partial_u\widehat{\phi} - uk^2\widehat{\phi} = \frac{2\kappa_{10}^2}{4\pi\Omega_4}\frac{g_s^2}{R^3}\frac{1}{\sqrt{U_T}}\delta(u-1)\,, \tag{4.102}$$

where the Fourier transform

$$\widehat{\phi}(u,k) = \int d^2k e^{i\vec{k}\vec{x}}\widehat{\phi}(x_1,x_2,u) \tag{4.103}$$

was introduced. The equation is similar to (4.88). Two solutions of the homogeneous eqution can again be defined by

$$\begin{aligned} y_1(u,k) &= u^{-3} + \mathcal{O}(u^{-4}) \qquad && u\to\infty \\ y_2(u,k) &= 1 + \mathcal{O}(u-1) \qquad && u \to 1 \end{aligned} \tag{4.104}$$

The Wronskian between these solutions is given by

$$W(y_1(u),y_2(u)) = y_1y_2' - y_2y_1' = \frac{w(k)}{u^4-u}\,, \tag{4.105}$$

where $w(k)$ is a function only of k and can be computed as a series expansion as was done in [46] or in a WKB approximation as we do in the Appendix. Now the field can be written as

$$\widehat{\phi}(u,k) = -\frac{2\kappa_{10}^2}{4\pi\Omega_4}\frac{g_s^2}{R^3}\frac{1}{\sqrt{U_T}}\frac{1}{w(k)}y_1(u,k)\,. \tag{4.106}$$

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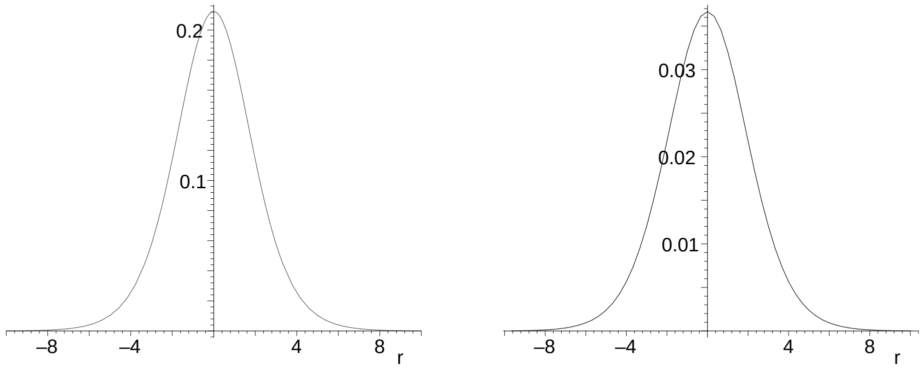

Figure 3: Profile of the QCD_{2+1} (left) and QCD_{3+1} (right) flux tube. The r-axis has units of the corresponding $1/m_{gl}$ and the profile is normalized so that its integral is 1.

Expanding this field near the boundary $u \to \infty$ and using the relation (4.43) with the expectation value of F^2, the Fourier transform of the string profile can be obtained as

$$\frac{1}{4g_{YM}^2}\langle F^2\rangle(k_1,k_2) = \frac{1}{4\pi}\left(\frac{U_T}{R}\right)^{\frac{3}{2}}\frac{3}{w(k)}\,. \tag{4.107}$$

Again one can check, using that $w(0) = 3$, the integral of this profile reproduces (4.84). The string profile is given by

$$\begin{aligned}\frac{1}{4g_{YM}^2}\langle F^2\rangle(k_1,k_2) &= \frac{1}{4\pi}\left(\frac{U_T}{R}\right)^{\frac{3}{2}}\int d^2k\frac{3}{w(k)}\\ &= \frac{1}{4\pi}\left(\frac{U_T}{R}\right)^{\frac{3}{2}}\frac{\pi^6}{\alpha^5}\int_0^\infty\frac{(y^4+y^2)J_0\left(\frac{\pi r}{\alpha}y\right)}{\sinh\pi y}dy\,,\end{aligned} \tag{4.108}$$

where $r^2 = x_1^2 + x_2^2$. The last integral cannot be obtained in closed form, but two expansions valid for large and small r are:

$$\begin{aligned}\int_0^\infty\frac{(y^4+y^2)J_0\left(\frac{\pi r}{\alpha}y\right)}{\sinh\pi y}dy &= \frac{2}{\pi}\sum_{n=2}^{\infty}(-)^n n^2(n^2-1)K_0\left(n\frac{\pi r}{\alpha}\right)\\ &= \frac{1}{(2\pi)^5}\sum_{n=0}^{\infty}c_n\frac{r^{2n}}{(4\alpha)^{2n}}\frac{(-)^n}{(n!)^2}\,,\end{aligned} \tag{4.109}$$

$$c_n = 2(2^{5+2n}-1)\Gamma(5+2n)\zeta(5+2n) + 8\pi^2(2^{3+2n}-1)\Gamma(3+2n)\zeta(3+2n)\,. \tag{4.110}$$

From the first series it follows that at large distances the profile decays exponentially as $\exp(-2\pi r/\alpha)$ which is right since $2\pi/\alpha$ is the mass of the lowest lying glueball in the approximation we are considering. The profile obtained is depicted in figure 3.

5. Conclusions

We have studied various aspects of the bulk/boundary correspondence. First, we addressed a basic issue about the possible vacuum choices in AdS space. Often one imposes the system to be in a Poincaré vacuum state, this vacuum is however defined with respect to a coordinate system which does not cover the whole manifold. Hence one might wonder whether it would be more natural to use the global vacuum state, which is defined with respect to the global coordinates covering the whole space. However, we showed that for practical purposes the two vacua can in fact be identified after subtracting off a constant zero point energy contribution to the vacuum.

We then turned our attention to various probes of the bulk and their description in the boundary theory. We started by an extension of an example considered in [22], a source in AdS space. Instead of the quasistatic treatment of [22], we took the source to be in free fall, and evaluated the projection of the source to the boundary by using retarded propagators instead of static propagators. This results in a slight modification of the results in [22]; instead of looking like a blob with a single scale, its size, the boundary profile looks like an expanding bubble characterized by two scales which encode the radial position and the state of motion of the source falling in the bulk.

Static sources are also possible, but in that case they should be kept fixed by *e.g.* suspending them on a string. The presence of the string is in turn represented by an additional point charge in the boundary, which induces a nonzero expectation value for the field strength, $\langle F^2 \rangle$. We evaluate this expectation value and find the correct form for the potential from the point charge.

At non-zero temperatures, the point charge is thermally screened. We examined this case as well, and evaluated the thermal screening length. It turned out to be given by glueball masses, of the order of the temperature T.

We also studied non-supersymmetric, confining QCD. In that case, the center of the (Euclidean) black hole plays a central role. The real world at the boundary has the character of a shadow of the world at the center of the Euclidean black hole. A string sitting in the center, appears in the boundary as a flux tube of finite width with a profile that can be computed. Pushing this picture a little further, a closed string near the center of the black hole is mapped into a glueball in the boundary. If the string state is light, then the center of mass will be spread around $U = U_T$ which is the case studied in [17]. One can also consider a qualitative picture of scattering between such glueballs.[8] If the bound states scatter at high energy, two distinct regimes appear. At small scattering angle the momentum transfer is small and so the particles in intermediate channels are the lowest lying glueballs. In that domain, when states far from the $U = U_T$ do not contribute, one expects a Regge behavior from the string scattering amplitudes which corresponds to the expected Regge behavior of glueball scattering in the boundary. At finite angles the momentum transfer is large and the region in which the bulk metric is AdS_5 is

[8]We thank Bo Sundborg for a discussion on this point.

expected to dominate. In the boundary that should correspond to the parton region in QCD, but unfortunately the theory studied at high energies is not QCD but $N = 4$ SYM, which is precisely what string theory in AdS_5 describes according to Maldacena's conjecture.

Acknowledgments

We would like to thank Ingemar Bengtsson, Per Kraus, Jorma Louko, and Bo Sundborg for discussions. The work of U.D. was supported by Swedish Natural Science Research Council (NFR) and that of M.K. by The Swedish Foundation for International Cooperation in Research and Higher Education (STINT).

A. Calculation of Wronskian through WKB

In this appendix we will compute, using a WKB approximation[9] the function $w(k)$ defined in eq. (4.69). To obtain a WKB solution of eq. (4.61) it is better to work with imaginary k and then analytically continue to the real axis. In that case, the equation (4.61) can be recast in the form of a Schrödinger equation for a zero-energy eigenstate [17]:

$$\begin{aligned} -\chi''(x) + V(x)\chi &= 0\,, \\ V(x) &= -\frac{k^2}{4x(x^2-1)} + \frac{3x^4 - 6x^2 - 1}{4x^2(x^2-1)^2}\,, \end{aligned} \tag{A.1}$$

where $x = u^2$ and $\chi(x) = \sqrt{x(x^2-1)}y(\sqrt{x})$. There are two regions where the solutions of interest can be expressed in terms of Bessel functions :

$$\begin{aligned} \chi_1(x,k) &= \sqrt{x(x^2-1)}y_1(\sqrt{x},k) \approx \frac{8\sqrt{x}}{k^2} J_2\left(\frac{k}{\sqrt{x}}\right), && \text{if } x \gg 1\,, \\ \chi_2(x,k) &= \sqrt{x(x^2-1)}y_2(\sqrt{x},k) \approx \sqrt{2(x-1)}J_0\left(k\sqrt{\frac{x-1}{2}}\right), && \text{if}(x-1) \ll 1\,. \end{aligned} \tag{A.2}$$

In the region $(x-1) \gg 1/k^2$, $x \ll k^2/3$ the potential is approximately given by

$$|V(x)| \approx \bar{V}(x) = \frac{k^2}{4x(x^2-1)}\,. \tag{A.3}$$

For large k we can use a WKB approximation:

$$\chi(x) \approx \frac{1}{|\bar{V}|^{\frac{1}{4}}}\left(Ae^{i\int_x^a \sqrt{\bar{V}(x)}} + Be^{-i\int_x^a \sqrt{\bar{V}(x)}}\right). \tag{A.4}$$

[9] while this paper was being written ref. [47] appears which has considerable overlap with what follows.

The strategy now is clear. For large k there is an overlap between the region $x-1 \gg 1/k^2$ and $x-1 \ll 1$, so we can compute the coefficients A_2 B_2 corresponding to the function χ_2. There is also an overlap between the regions $x \gg 1$ and $x \ll k^2/3$, so we can compute the coefficients A_1, B_1 corresponding to χ_1. Finally, the Wronskian between the two WKB solutions can be computed. The coefficients are given by

$$A_1 = B_1^* = \frac{4}{\sqrt{\pi}k^2} e^{-\frac{5}{4}i\pi + i\frac{k}{\sqrt{a}}}, \tag{A.5}$$

$$A_2 = B_2^* = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{4}i\pi - ik\sqrt{\frac{b-1}{2}} + i\int_b^a \sqrt{\bar{V}(x)}}, \tag{A.6}$$

where we assume that the two points a, b are such that $1 \gg a-1 \gg 1/k^2$ and $1 \ll b \ll k^2/3$. The Wronskian can now be obtained as

$$\begin{aligned} W(\chi_1,\chi_2) &= \frac{w(k)}{2} = 2i(A_1B_2 - A_2B_1) \\ &= -\frac{8\sqrt{2}}{\pi k^2}\cos\left(\frac{k}{2}\int_a^b \frac{dx}{\sqrt{x(x^2-1)}} + \frac{k}{\sqrt{a}} + k\sqrt{\frac{b-1}{2}}\right). \end{aligned} \tag{A.7}$$

The argument of the cosine is approximately independent of a and b, so we can take $b=1$, $a=\infty$. The result is

$$w(k) = -\frac{16\sqrt{2}}{\pi k^2}\cos(\alpha k), \tag{A.8}$$

$$\alpha = \frac{1}{2}\int_1^\infty \frac{dx}{\sqrt{x(x^2-1)}} = \frac{1}{4\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2. \tag{A.9}$$

From here the masses of the 0^{++} glueballs follow immediately as

$$m = \frac{\pi}{\alpha}\left(n+\frac{1}{2}\right) \tag{A.10}$$

The obtained values are in good agreement[10] with the numerical ones except for the fact that there is an extra eigenvalue at $n=0$. Of course this is at small k, i.e. outside the validity of the approximation. However a simple modification of the Wronskian cures this problem and gives a good approximation in the full range $k = 0 \to \infty$ for the function $w(k)$ obtained numerically. The modification is to write

$$w(k) = -\frac{16\sqrt{2}}{\pi\left(k^2 - \left(\frac{\pi}{2\alpha}\right)^2\right)}\cos(\alpha k). \tag{A.11}$$

[10] As it turns out, the values of [47] are even better since there a next-to-leading order was considered

In this approximation, $w(0) \approx 5.017$ to be compared with the exact value $w(0) = 4$. Analytically continuing $k \to ik$ we obtain the desired function

$$w(k) = \frac{16\sqrt{2}}{\pi\left(k^2 + \left(\frac{\pi}{2\alpha}\right)^2\right)} \cosh(\alpha k)\,. \tag{A.12}$$

The other case studied in the text corresponds to the homogeneous equation

$$(u^4 - u)y''(u) + (4u^3 - 1)y'(u) - uk^2 y(u) = 0\,. \tag{A.13}$$

The quantity of interest is the Wronskian between the two solutions defined by

$$\begin{aligned} y_1(u,k) &= \frac{1}{u^3} + \mathcal{O}(u^{-4}) \qquad && u \to \infty \\ y_2(u,k) &= 1 + \mathcal{O}(u-1) && u \to 1 \end{aligned} \tag{A.14}$$

To perform a WKB analysis we change variables $x = \sqrt{u}$ and $\chi = \sqrt{x^7 - x}y$. The resulting equation is of the Schrödinger type

$$-\chi''(x) + V(x)\chi = 0\,, \tag{A.15}$$

$$V(x) = \frac{1}{4}\frac{35x^{12} - 70x^6 - 1}{x^2(x^6-1)^2} - \frac{4x^3k^2}{x^7 - x}\,. \tag{A.16}$$

The WKB analysis is completely similar to the previous case. The result is

$$W(y_1(u), y_2(u)) = \frac{w(k)}{u^4 - u}\,, \tag{A.17}$$

$$w(k) = -\frac{6\sqrt{3}}{\pi k^3}\sin(\alpha k)\,, \tag{A.18}$$

with

$$\alpha = \int_1^\infty \frac{dy}{\sqrt{y^3 - 1}} = \frac{\sqrt{\pi}}{3}\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{2}{3}\right)}\,. \tag{A.19}$$

There is also a pair of extra poles at $k = \pm\pi/\alpha$. This can be cured replacing the Wronskian by

$$w(k) = -\frac{6\sqrt{3}}{\pi k}\frac{1}{k^2 - \frac{\pi^2}{\alpha^2}}\sin(\alpha k)\,. \tag{A.20}$$

From where we read the glueball masses as

$$m_n = \frac{\pi}{\alpha}n, \quad n = 2, 3, \ldots \tag{A.21}$$

in good agreement with the results of [17]. Analytically continuing to $k \to ik$ the desired Wronskian is

$$w(k) = \frac{6\sqrt{3}}{\pi k}\frac{1}{k^2 + \frac{\pi^2}{\alpha^2}}\sinh(\alpha k)\,. \tag{A.22}$$

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